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Analysis of Correlation in the Intercomparison of DC Voltage Reference Standards

Mauricio Sáchica and Alexander Martínez

***Abstract*— This paper describes the application of GUM supplement 1 and supplement 2 in the intercomparison procedure of a group of four dc voltage reference standards. The results obtained are presented and according to these, the choice of a restriction is suggested to improve the uncertainty and avoiding the strong correlation of the estimated values.**

***Index Terms*— Correlation, least-squares method, measurement, Monte Carlo simulation, uncertainty.**

I. Introduction

THE intercomparison of dc voltage reference standards by opposition method is used to verify the stability of their values, by means of the measurement of all possible differences y_i between them and the use of a system constraint.

The results obtained in the intercomparison of dc voltage reference standards will always be correlated due to the system model. The influence on the correlation by the constraint used is analyzed in this paper.

For each pair of reference standards, measurements are taken using "left-right" balance [1] to cancel the residual electromotive force that remain approximately constant

$$\begin{aligned} y_1 &= V_1 - V_2 \\ y_2 &= V_2 - V_1 \\ y_3 &= V_1 - V_3 \\ y_4 &= V_3 - V_1 \\ y_5 &= V_1 - V_4 \\ y_6 &= V_4 - V_1 \\ y_7 &= V_2 - V_3 \\ y_8 &= V_3 - V_2 \\ y_9 &= V_2 - V_4 \\ y_{10} &= V_4 - V_2 \\ y_{11} &= V_3 - V_4 \\ y_{12} &= V_4 - V_3. \end{aligned} \tag{1}$$

As shown above, there is an overdetermined system of equations that requires a constraint for its solution by least-squares method [2].

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The authors are with the Instituto Nacional de Metrología, Bogotá 111321, Colombia (e-mail: msachica@inm.gov.co; amartinez@inm.gov.co).

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As will be discussed along this paper, the correlation and the uncertainties of the estimated values, with which the behavior of the references is verified, will depend on the constraint used.

Considering the constraint as follows:

$$V_{\text{sum}} = V_{1P} + V_{2P} + V_{3P} + V_{4P}. \tag{2}$$

The values V_{1P}, V_{2P}, V_{3P}, and V_{4P} used to find the constraint could be assigned in three different ways.

1) The certified values of the dc voltage reference standards according to their last calibration.
2) The expected values of the dc voltage reference standards at the date of intercomparison.
3) The nominal values of the dc voltage reference standards.

The matrix equation of the system is

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \\ \hat{V}_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ V_{\text{sum}} \end{bmatrix} \tag{3}$$

$$K \cdot \hat{V} = Y. \tag{4}$$

The solution of the system, corresponding to the estimated values in the verification, is defined as

$$\hat{V} = (K^t \cdot K)^{-1} \cdot K^t \cdot Y. \tag{5}$$

The variation of the estimated values $\hat{V}$ is analyzed using control charts to determine the behavior of the dc voltage reference standards.

II. Choosing the Constraint

According to [1], for a group of saturated standard cells, the system constraint corresponded to the average of the assigned values obtained of their last calibration; however, this is not applicable for our group of dc voltage reference standards. As shown in Fig. 1, the four reference standards have negative drift, so the average of the values will not remain constant and will tend to decrease.

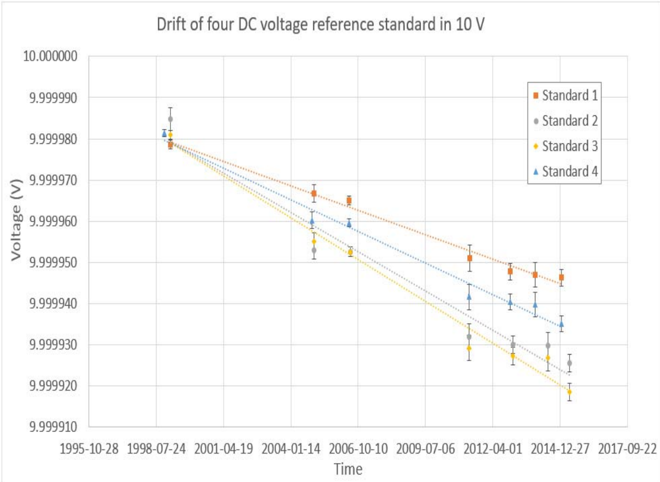

Fig. 1. Drift of four dc voltage reference standard in 10 V.

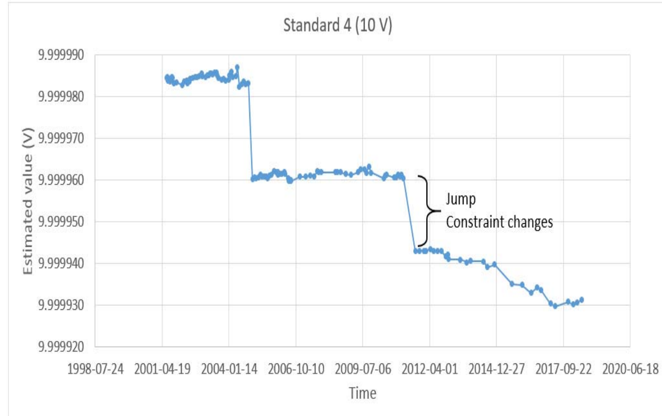

Fig. 2. Behavior of estimated value of a dc voltage reference standard resulting from intercomparison method using the last calibration values of the dc voltage reference standards as restriction.

The drifts presented in Fig. 1 were obtained using the certified values of the four dc voltage reference standards.

The traceability of these values is obtained from two of the standards that are calibrated externally in CENAM and PTB.

If we use this constraint assuming that the average of the certified values of the dc voltage reference standards remains constant until the next calibration, the control charts, that allow verifying their behavior, will show jumps each time these dc voltage reference standards are calibrated due to constraint changes (Fig. 2).

As could be evidenced, the average of the values is not maintained over time, so the constraint could be assigned as the sum of the predicted values of the reference standards from their historical certified values, these predicted values correspond to the expected values at the date of intercomparison.

In this way, we obtain the estimated values of the dc voltage reference standards $\hat{V}$ from the measured differences and the predicted values according to their previous calibrations. Using this constraint, we will not have the jumps that occur when we assume that the average of the reference standards remains constant between calibrations since in each intercomparison the predicted drift will be taken into account for the average of the dc voltage reference standards

A disadvantage of considering the constraint either as a constant average of the dc voltage reference standards, or as the sum of their predicted values, is that they depend on certified values, and therefore the uncertainty due to their calibration process must be considered on the final intercomparison uncertainty, so the estimated values by the intercomparison process, with which we will verify the behavior of the reference standards, will have uncertainties greater than their calibration and inadequate to perform the verification of the dc voltage reference standards.

It is more convenient to use a constant constraint, such as the sum of nominal values of the dc voltage reference standards,

$$
\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ V_{\text{sum}} \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} & V_{\text{sum}} \end{array} \\
\left[\begin{array}{ccccccccccccc}
1 & -1 & 0.5 & -0.5 & 0.5 & -0.5 & -0.5 & 0.5 & -0.5 & 0.5 & 0 & 0 & 0 \\
-1 & 1 & -0.5 & 0.5 & -0.5 & 0.5 & 0.5 & -0.5 & 0.5 & -0.5 & 0 & 0 & 0 \\
0.5 & -0.5 & 1 & -1 & 0.5 & -0.5 & 0.5 & -0.5 & 0 & 0 & -0.5 & 0.5 & 0 \\
-0.5 & 0.5 & -1 & 1 & -0.5 & 0.5 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 \\
0.5 & -0.5 & 0.5 & -0.5 & 1 & -1 & 0 & 0 & 0.5 & -0.5 & 0.5 & -0.5 & 0 \\
-0.5 & 0.5 & -0.5 & 0.5 & -1 & 1 & 0 & 0 & -0.5 & 0.5 & -0.5 & 0.5 & 0 \\
-0.5 & 0.5 & 0.5 & -0.5 & 0 & 0 & 1 & -1 & 0.5 & -0.5 & -0.5 & 0.5 & 0 \\
0.5 & -0.5 & -0.5 & 0.5 & 0 & 0 & -1 & 1 & -0.5 & 0.5 & 0.5 & -0.5 & 0 \\
-0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0.5 & -0.5 & 1 & -1 & 0.5 & -0.5 & 0 \\
0.5 & -0.5 & 0 & 0 & -0.5 & 0.5 & -0.5 & 0.5 & -1 & 1 & -0.5 & 0.5 & 0 \\
0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 & -0.5 & 0.5 & 0.5 & -0.5 & 1 & -1 & 0 \\
0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 & 0.5 & -0.5 & -0.5 & 0.5 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array} \tag{6}
$$

$$
\text{cov}(Y) = \begin{bmatrix}
u^2(y_1) & u(y_1)u(y_2)r(y_1, y_2) & \dots & u(y_1)u(V_{\text{sum}})r(y_1, V_{\text{sum}}) \\
u(y_2)u(y_1)r(y_2, y_1) & u^2(y_2) & \dots & u(y_2)u(V_{\text{sum}})r(y_2, V_{\text{sum}}) \\
\vdots & \vdots & \ddots & \vdots \\
u(y_{12})u(y_1)r(y_{12}, y_1) & u(y_{12})u(y_2)r(y_{12}, y_2) & \dots & u(y_{12})u(V_{\text{sum}})r(y_{12}, V_{\text{sum}}) \\
u(V_{\text{sum}})u(y_1)r(V_{\text{sum}}, y_1) & u(V_{\text{sum}})u(y_2)r(V_{\text{sum}}, y_2) & \dots & u^2(V_{\text{sum}})
\end{bmatrix} \tag{7}
$$

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so there is no traceability component, and therefore no values would be assigned to the standards.

The values obtained from the intercomparison method using a constant constraint would only be control values that provide information about the dc voltage reference standards behavior with an optimal uncertainty.

The uncertainty of the estimated values $\hat{V}$ is estimated according to n-dimensional models [3] and Monte Carlo simulation [4]. The correlation of the input measurements y_i is analyzed and included in the uncertainty for the estimated values $\hat{V}$, the correlation between these values is then obtained.

With the estimated values $\hat{V}$ and their uncertainty, the behavior of the voltage references in the time can be verified.

III. Correlation of the Input Measurements

The correlation of the measurements y_i can be obtained from the models in (1). The correlation coefficients of all the possible pairs combinations $r(y_i, y_j)$ can be calculated taking into account the common values V_1, V_2, V_3, and V_4 between the different y_i.

To find the correlation coefficients $r(y_i, y_j)$, it was assumed that the values of the voltage reference standards follow a probability distribution with the same standard deviation.

Therefore, the correlation matrix r from the input measurements y_i is given in (6).

The correlation matrix (6), as shown at the bottom of the previous page, allows quantifying the functional correlation between the input measurements which depends on the voltage reference standards common to y_i and y_j.

IV. Estimation of Uncertainty

A. n-Dimensional Models "Supplement 2 to the GUM"

With the correlation matrix r, the uncertainty of measurements $u(y_i)$ and the uncertainty of the constraint $u(V_{\text{sum}})$, we can obtain the respective covariance matrix from (7), as shown at the bottom of the previous page.

Considering $F = (K^t \cdot K)^{-1} \cdot K^t$ and taking into account the residuals ε by the least-squares adjustment for the system solution, the measurement model would be

$$\hat{V} = F \cdot (Y + \varepsilon). \tag{8}$$

The matrix covariance of $\hat{V}$ is

$$\text{cov}(\hat{V}) = s_e^2 \cdot (K^t \cdot K)^{-1} + F \cdot \text{cov}(Y) \cdot F^t \tag{9}$$

where s_e^2 corresponds to the variance associated with the fitting of the solution by least-squares method and is defined as follows:

$$s_e^2 = \frac{(K \cdot \hat{V} - Y)^t (K \cdot \hat{V} - Y)}{m - (p+1)} \tag{10}$$

where $m = 13$ system equations and $p = 4$ output quantities.

B. Monte Carlo Simulation "Supplement 1 to the GUM"

To perform the Monte Carlo simulation, GNU Octave programing language is used. Each of the inputs y_i and V_{sum} is simulated as follows.

1) Multivariate standard normal independent distributions are simulated for y_i and V_{sum} with 10^6 trials.
2) The transformation matrix R is found to include the correlation in the simulated distributions, performing a decomposition of Cholesky to the matrix $\text{Cov}(Y)$. In order to perform the Cholesky decomposition, $\text{Cov}(Y)$ matrix must be positive defined, which for this case is not, therefore a repair algorithm [5] must be applied to this matrix.
3) The transformation of the standard normal distributions is applied so that they have the required mean, variance, and correlation coefficient according to previous information and measurements.

With the simulation of Y, the following models derived from (5) are applied to find the system output distributions for $\hat{V}$

$$\begin{aligned}
\hat{V}_1 &= \frac{V_{\text{sum}}}{4} + \frac{1}{8}(y_1 - y_2 + y_3 - y_4 + y_5 - y_6) \\
\hat{V}_2 &= \frac{V_{\text{sum}}}{4} + \frac{1}{8}(-y_1 + y_2 + y_7 - y_8 + y_9 - y_{10}) \\
\hat{V}_3 &= \frac{V_{\text{sum}}}{4} + \frac{1}{8}(-y_3 + y_4 - y_7 + y_8 + y_{11} - y_{12}) \\
\hat{V}_4 &= \frac{V_{\text{sum}}}{4} + \frac{1}{8}(-y_5 + y_6 - y_9 + y_{10} - y_{11} + y_{12}).
\end{aligned} \tag{11}$$

V. Influence of Correlation on Intercomparison Results

As shown in (11), the constraint V_{sum} is common to all estimated values $\hat{V}_i$ and those values also share input measurements y_i, so the estimated values $\hat{V}$ are correlated.

The estimated values of the dc voltage reference standards resulting from the intercomparison process are completely correlated when the constraint used is either the sum of the standards certified values or the sum their predicted values. This correlation was verified with the correlation matrix of $\hat{V}$ found by the n-dimensional evaluation shown in the following:

$$\begin{array}{c} \\ \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \\ \hat{V}_4 \end{array}
\begin{array}{c} \begin{array}{cccc} \hat{V}_1 & \hat{V}_2 & \hat{V}_3 & \hat{V}_4 \end{array} \\
\begin{bmatrix} 1.0000 & 0.9994 & 0.9995 & 0.9994 \\ 0.9994 & 1.0000 & 0.9995 & 0.9995 \\ 0.9995 & 0.9995 & 1.0000 & 0.9995 \\ 0.9994 & 0.9995 & 0.9995 & 1.0000 \end{bmatrix} \end{array}. \tag{12}$$

Correlation can also be evidenced in Fig. 3, in which the Monte Carlo simulation was used.

The estimated values $\hat{V}$ of the voltage reference standards resulting from the intercomparison process are completely correlated. This largely due to the system constraint V_{sum}.

In this case, $u(V_{\text{sum}})$ is significantly greater than the uncertainties of the measurements $u(y_i)$.

But if the sum of nominal values of the voltage reference standards is used as the constraint V_{sum}, the estimated values

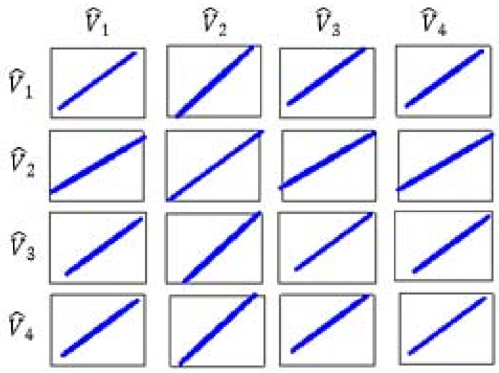

Fig. 3. Graphic correlation matrix of estimated values $\hat{V}$. Monte Carlo simulation was used.

correlation is not significant. This was verified with the correlation matrix of $\hat{V}$ found by the n-dimensional evaluation shown in the following:

$$\begin{array}{c} \\ \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \\ \hat{V}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \hat{V}_1 & \hat{V}_2 & \hat{V}_3 & \hat{V}_4 \end{array} \\ \begin{bmatrix} 1.0000 & 0.0418 & 0.0815 & 0.0415 \\ 0.0418 & 1.0000 & 0.0287 & 0.0690 \\ 0.0815 & 0.0287 & 1.0000 & 0.0636 \\ 0.0415 & 0.0690 & 0.0636 & 1.0000 \end{bmatrix} \end{array}. \tag{13}$$

As it was evidenced, the correlation of the estimated values resulting from the intercomparison process is strongly influenced by the system constraint V_{sum}.

VI. Analysis of Results

According to the intercomparison model, the correlation will always exist, but it is strongly influenced by the system constraint V_{sum}.

The matrix of correlation r (6) is ideal and was found theoretically from the system of equations with some system characteristics assumed. We could see that in the real exercise there are some variations of this matrix; however, it was identified that changes in the correlation between the y_i does not affect significantly the estimated values nor their uncertainty.

When the constraint V_{sum} is assigned from the certified or predicted values of the standards, $u(V_{\text{sum}})$ is the dominant uncertainty component in the intercomparison process. There is a previous correlation between the certified or predicted values of the standards since they are assigned from the same reference standard in the calibration process. This previous correlation must be included in $u(V_{\text{sum}})$, its omission generates an underestimation of the intercomparison estimated values uncertainty.

The uncertainties for the estimated values $\hat{V}$ depend on the assigned constraint V_{sum}. If these uncertainties are greater than the calibration uncertainty of the standards, they are inadequate to perform their verification. To obtain an optimal uncertainty and avoid strong correlation in the final results, the constant constraint is the most convenient one.

This paper is based on the summary presented at the Conference on Precision Electromagnetic Measurements CPEM 2018 [6].

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Mauricio Sáchica was born in Bogotá, Colombia, in 1987. He received the B.Eng. degree in electronics from the Universidad Distrital Francisco José de Caldas, Bogotá, in 2009, and the Specialist degree in industrial process automation from the Universidad de los Andes, Bogotá, in 2012.

In 2010, he joined the Metrology Group, the Industry and Commerce Superintendence, Bogotá (reorganized into the National Metrology Institute of Colombia, Bogotá, in 2011). In 2011 and 2012, he developed automation systems in hydrocarbon pumping stations. Since 2013, he has been with the National Metrology Institute of Colombia, where he has been involved in electrical metrology.

Alexander Martínez was born in Bogotá, Colombia, in 1966. He received the B.S. degree in physics from the National University of Colombia, Bogotá, in 1993.

In 1994 and 2012, he joined the Metrology Group, Industry and Commerce Superintendence, Bogotá, where he was responsible for establishing the national voltage and resistance standards and involved in metrology and measurement uncertainty. In 2012, he joined the National Metrology Institute of Colombia(INM), Bogotá, where he is the Coordinator of the laboratories of electrical metrology, temperature and humidity. He is the Second Leader of the INM Research Group in the development of measurement standards.